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ON THE NILPOTENCY IN MATRIX ALGEBRAS WITH GRASSMANN ENTRIES

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The paper is dedicated to the 65th birthday of Prof. Yuri A. Bahturin

Abstract. In the paper we consider some classes of subalgebras of $M_n(E)$ (for a given n and any n) for E being the Grassmann algebra. We give an estimation of the index of nilpotency of the commutators of length 2 for these classes of matrix algebras.

1. Preliminaries. In the paper we work with the infinite dimensional Grassmann algebra E and the finite dimensional Grassmann algebras $E_k = E(V_k)$ for arbitrary k. The algebra E is defined as

$$E = E(V) = K\langle e_1, e_2, \dots \mid e_ie_j + e_je_i = 0 \ i, j = 1, 2, \dots\rangle.$$

The field K has characteristic zero. If $\dim V = k$ the corresponding finite dimensional algebra $E(V_k)$ will be denoted by E_k.

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The elements of the algebras E and E_k will be called Grassmann elements. The number s defines the degree of the element $e_{i_1} \cdots e_{i_s}$.

The algebra E is in the mainstream of recent research in PI-theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer. In [8, Theorem 1.2] he proved that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(E)$ and $M_{n,u}(E)$, the latter being the algebra of $n \times n$ supermatrices over $E = E_0 \oplus E_1$ with two E_0 blocks (with entries of even degree) of sizes $u \times u$ and $(n-u) \times (n-u)$ and with two E_1 blocks (with entries of odd degree) of sizes $u \times (n-u)$ and $(n-u) \times u$.

Another reason for the Grassmann algebra to be one of the fundamental structures in PI-theory is the fact that it generates a minimal variety of exponential growth [9].

There is a motivation of considering finite-dimensional Grassmann algebras as well and it is connected with the emergence in mathematical physics mainly in quantum field theory of methods from algebraic geometry and Grassmann algebras. We give only three examples here:

If we take a Grassmann algebra with two generators y and y^* and a conjugation $*$ we have $(y^*)^* = y$ and one could define Grassmann differentiation and integration, the exponential function, scalar product of linear functions, etc. Thus when studying fundamental particles in nature one can describe with the special features of the Grassmann algebra Fermion coherent states analogously to Boson coherent states and thus to investigate its physical significance [5].

In [12] Schornhorst considers a special type integral equation with an unknown function over a finite dimensional Grassmann algebra E_{2n} and gives conditions for the existence of solutions of this equation for $n = 2$ and $n = 4$. The choice of the equation is motivated by the effective action formalism of lattice quantum field theory.

In a Grassmann variant of classical mechanics functions on phase space in nonrelativistic theory are elements of a Grassmann algebra with three generators [2].

The importance of considering matrix algebras $M_n(E)$ is confirmed by the following statement as the trivial isomorhism $E \otimes M_n(K) \simeq M_n(E)$ holds:

Proposition 1 [4, Corollary 8.2.4, p. 111]. *For every PI-algebra R there exists a positive n such that $T(R) \supseteq T(M_n(E))$, i.e. R satisfies all polynomial identities of the $n \times n$ matrix algebra $M_n(E)$ with entries from the Grassmann algebra.*

We list some well known facts concerning both the algebras E and $M_n(E)$ using [9, 3, 1].

Proposition 2 [9, Corollary, p. 437]**.** *The T-ideal $T(E)$ is generated by the identity $[x_1, x_2, x_3] = 0$.*

Proposition 3 [3, Lemma 5.1]**.** *The T-ideal $T(E)$ contains the identities $[x_1, x_2][x_2, x_3] = 0$ and $[x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4] = 0$.*

Proposition 4 [1, Lemma 6.1]**.** *The algebra E satisfies $S_n(x_1, \ldots, x_n)^k = 0$ for all $n, k \geq 2$ and*

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma \in \mathrm{Sym}(n)} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

being the standard identity.

Proposition 5 [1, Corollary 6.6]**.** *The algebra $M_n(E)$ does not satisfy the identity $S_m(x_1, \ldots, x_m)^n = 0$ for any m.*

Proposition 6 [3, Exercise 5.3]**.** *For $E_k = E(V_k)$ over k-dimensional vector space V_k all identities follow from the identity $[x_1, x_2, x_3] = 0$ and the standard identity*

$$S_{2p}(x_1, \ldots, x_{2p}) = 0,$$

where p is the minimal integer such that $2p > k$.

Proposition 7 [6, Theorem 3.5]**.** *Let K be an infinite field. A basis of the identities of E_{2k} is given by the polynomials*

$$[x_1, x_2, x_3] = 0, \;\; [x_1, x_2] \cdots [x_{2k+1}, x_{2k+2}] = 0.$$

Proposition 8 [11, Theorem]**.** *The matrix algebra $M_n(E)$ has no identities of degree $4n - 2$.*

Vishne gave in [14] explicit identities of degree 8 for $M_2(E)$ and concluded the following

Proposition 9 [14, Corollary 4.5]. *If n is even the degree of a multilinear identity for $M_n(E)$ is at least $4n$.*

The identity of "algebraicity" for matrices over the Grassmann algebra was defined by J. Szigeti in [13].

Proposition 10 [13, Theorem 5.1.]. *The polynomial*

$$S_{2n^2}([X^{2n^2}, Y], [X^{2n^2-1}, Y], \dots, [X^2, Y], [X, Y]) = 0$$

is an identity for $M_n(E)$.

More facts concerning the PI-structure of E and $M_n(E)$ could be found in [7].

2. Some identities for finite Grassmann algebras and corollaries.

Both Proposition 3 and Proposition 4 give that $(xy - yx)^2 = 0$ holds in any Grassmann algebra. We could formulate the following generalization, namely

Theorem 1. *The algebra E_k satisfies the identities*

$$(x_1 x_2 \cdots x_s - x_s x_{s-1} \cdots x_1)^n = 0$$

for all $n > k/2$.

Proof. For $s = 2$ we apply Proposition 4 and $n \geq 2$.

We state that the polynomial $x_1x_2 - x_2x_1$ as a Grassmann element for any x_1 and x_2 is of degree ≥ 2, the same for $x_1x_2x_3 - x_2x_1x_3$. Obviously $x_2x_1x_3 - x_3(x_2x_1)$ is of degree ≥ 2 and the same is valid for the sum $x_1x_2x_3 - x_2x_1x_3 + x_2x_1x_3 - x_3(x_2x_1) = x_1x_2x_3 - x_3x_2x_1$. The Grassmann element $(x_1x_2x_3 - x_3x_2x_1)^2$ is of degree ≥ 4. Thus in $(x_1x_2x_3 - x_3x_2x_1)^n$ the terms are of length $\geq 2n$ and $(x_1x_2x_3 - x_3x_2x_1)^n = 0$ for $2n > k$.

Analogously we get that $x_1x_2x_3x_4 - x_3x_2x_1x_4$ and $x_3x_2x_1x_4 - x_4(x_3x_2x_1)$ are of degree ≥ 2. Thus $x_1x_2x_3x_4 - x_4x_3x_2x_1$ is of degree ≥ 2, $(x_1x_2x_3x_4 - x_4x_3x_2x_1)^n$ has terms of length $\geq 2n$ and $(x_1x_2x_3x_4 - x_4x_3x_2x_1)^n = 0$ for $2n > k$.

Consequently s in Theorem 1 could be any natural number. □

As Proposition 5 states the algebra $M_2(E)$ does not satisfy $S_n^2 = 0$ for any n i.e. $M_2(E)$ does not satisfy $[x, y]^2 = 0$. Considering however some special matrices x and y of $M_2(E)$ we could prove the nilpotency of $[x, y]$.

Proposition 11. *The commutator of two symmetric matrices A_1 and A_2 of type $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ in $M_2(E)$ is nilpotent with index of nilpotency ≤ 3.*

Proof. Let $A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & \alpha_i \end{pmatrix}$ $(i = 1, 2)$. Thus

$$[A_1, A_2] = \begin{pmatrix} [\alpha_1, \alpha_2] + [\beta_1, \beta_2] & [\alpha_1, \beta_2] - [\alpha_2, \beta_1] \\ [\alpha_1, \beta_2] - [\alpha_2, \beta_1] & [\alpha_1, \alpha_2] + [\beta_1, \beta_2] \end{pmatrix}.$$

Then we consider $[A_1, A_2]^2$. Using the fact that in G commutators commute and Proposition 3 we get that $[A_1, A_2]^2$ is equal to the matrix

$$2 \begin{pmatrix} [\alpha_1, \alpha_2][\beta_1, \beta_2] - [\alpha_1, \beta_2][\alpha_2, \beta_1] & 0 \\ 0 & [\alpha_1, \alpha_2][\beta_1, \beta_2] - [\alpha_1, \beta_2][\alpha_2, \beta_1] \end{pmatrix}.$$

The commutative multiplication of commutators and Proposition 4 give that $[A_1, A_2]^3 = 0$. $\square$

Proposition 11 holds as well for any two matrices of type $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ in $M_2(E)$.

Corollary 1. *The commutator of any two matrices in $M_n(E)$ either of type*

$$\begin{pmatrix} \alpha_i & 0 & \dots & 0 & \beta_i \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_i & 0 & \dots & 0 & \alpha_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha_i & 0 & \dots & 0 & \beta_i \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta_i & 0 & \dots & 0 & \alpha_i \end{pmatrix}$$

is nilpotent with index of nilpotency ≤ 3.

We could generalize Proposition 11 for 3×3 matrices.

Proposition 12. *The commutator of two symmetric matrices A_1 and A_2 of type $\begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}$ in $M_3(E)$ is nilpotent with index of nilpotency ≤ 3.*

The commutator of any two matrices of type $\begin{pmatrix} \alpha & \beta & \beta \\ -\beta & \alpha & \beta \\ -\beta & -\beta & \alpha \end{pmatrix}$ *in* $M_3(E)$ *is nilpotent with index of nilpotency* ≤ 3.

P r o o f. We give the considerations for the symmetric matrices only. The entries of the commutator $[A_1, A_2] = [a_{ij}]$ are

$$a_{11} = a_{22} = a_{33} = [\alpha_1, \alpha_2] + 2[\beta_1, \beta_2]$$

and all others are equal to $[\alpha_1, \beta_2] + [\beta_1, \alpha_2] + [\beta_1, \beta_2]$.

Evaluating $[A_1, A_2]^2 = [b_{ij}]$ we get that all entries b_{ij} are of type

$$c_{ij}([\alpha_1, \alpha_2][\beta_1, \beta_2] + [\alpha_1, \beta_2][\beta_1, \alpha_2]),\ c_{ij} \in K.$$

The commutativity of commutators' multiplication, Proposition 3 and Proposition 4 applied to the entries of the matrix $[A_1, A_2]^3$ end the proof. $\square$

Remark 1. It is easy to be seen that Proposition 12 holds for $n \times n$ matrices as well.

P r o o f. In this case for the matrix $[A_1, A_2] = [a_{ij}]$ we have

$$\begin{aligned} a_{ii} &= [\alpha_1, \alpha_2] + (n-1)[\beta_1, \beta_2], \\ a_{ij} &= [\alpha_1, \beta_2] + [\beta_1, \alpha_2] + (n-2)[\beta_1, \beta_2]. \end{aligned}$$

The elements b_{ij} of $[A_1, A_2]^2$ are of type

$$c_{ij}[\alpha_1, \alpha_2][\beta_1, \beta_2] + d_{ij}[\alpha_1, \beta_2][\beta_1, \alpha_2]$$

for $c_{ij}, d_{ij} \in K$. $\square$

Proposition 4 and a result of Giambruno and Zeicev [7, Theorem 1.9.1] give that the commutator of any two upper triangular matrices from $U_n(E)$ is nilpotent of index $\leq 2n$. Here we give examples with better estimation of the index of nilpotency. Some of them are from a list of algebras M_i with involution $*$ considered in [10], where the clasification of *-varieties whose sequence of $*$-codimensions c_n^* is linearly bounded is given. For example the algebra M_1 considered in [10] for which $c_n^*(M_1) = 1 + n + \dfrac{n(n-1)}{2}$ is a subalgebra of $U4(E)$ (defined below in the text as all $Ui(E)$). The subalgebra $U9(E)$ for $n = 4$ is a subalgebra of M_7 and for $n = 3$ is a subalgebra of M_4. The algebra M_6 is a subalgebra of $U10(E)$ for $n = 4$. As proved in [10] $c_n^*(M_4) = c_n^*(M_6) = c_n^*(M_7) \geq n(n-2)$ for all $n \geq 3$.

Proposition 13. *All matrices of the subalgebras*

$$U1(E) = \left\{ \begin{pmatrix} \alpha & \delta & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right\}, \quad U2(E) = \left\{ \begin{pmatrix} \alpha & 0 & \delta \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right\} \text{ and}$$

$$U3(E) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \delta \\ 0 & 0 & \gamma \end{pmatrix} \right\}$$

have nilpotent commutators with index of nilpotency ≤ 4.

Proof. We give the considerations only for $U3(E)$. In the other cases they are analogous ones. For $A_i = \begin{pmatrix} \alpha_i & 0 & 0 \\ 0 & \beta_i & \delta_i \\ 0 & 0 & \gamma_i \end{pmatrix}$, $i = 1, 2$ we get that

$$[A_1, A_2] = \begin{pmatrix} [\alpha_1, \alpha_2] & 0 & 0 \\ 0 & [\beta_1, \beta_2] & \beta_1\delta_2 + \delta_1\gamma_2 - \beta_2\delta_1 - \delta_2\gamma_1 \\ 0 & 0 & [\gamma_1, \gamma_2] \end{pmatrix}.$$

The only nonzero entry of $[A_1, A_2]^2 = [a_{ij}]$ is a_{23} and thus we get that $[A_1, A_2]^4 = 0$. $\square$

Proposition 14. *All matrices of the subalgebras*

$$U4(E) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \right\} \quad \text{and} \quad U5(E) = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

have nilpotent commutators with index of nilpotency ≤ 3.

Proof. Considering $U4(E)$ for $A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ 0 & \alpha_i & \delta_i \\ 0 & 0 & \alpha_i \end{pmatrix}$, $i = 1, 2$ we get that

$$[A_1, A_2] = \begin{pmatrix} [\alpha_1, \alpha_2] & [\alpha_1, \beta_2] + [\beta_1, \alpha_2] & [\alpha_1, \gamma_2] + [\gamma_1, \alpha_2] + \beta_1\delta_2 - \beta_2\delta_1 \\ 0 & [\alpha_1, \alpha_2] & [\alpha_1, \delta_2] + [\delta_1, \alpha_2] \\ 0 & 0 & [\alpha_1, \alpha_2] \end{pmatrix}.$$

The only nonzero entry of $[A_1, A_2]^2 = [a_{ij}]$ is

$$a_{13} = 2[\alpha_1, \alpha_2](\beta_1\delta_2 - \beta_2\delta_1) + [\alpha_1, \beta_2][\delta_1, \alpha_2] + [\beta_1, \alpha_2][\alpha_1, \delta_2].$$

Using Proposition 3 and the commuting of commutators we see that $[A_1, A_2]^3 = 0$.

Analogous are the considerations for $U5(E)$. In this case for $[B_1, B_2]^2 = [b_{ij}]$ the only nonzero entry is

$$b_{13} = [\alpha_1, \alpha_2](\alpha_1\gamma_2 + \beta_1\delta_2 - \alpha_2\gamma_1 - \beta_2\delta_1) + ([\alpha_1, \beta_2] + [\beta_1, \alpha_2])(\alpha_1\delta_2 - \alpha_2\delta_1). \quad \square$$

Proposition 15. *All matrices of the subalgebras*

$$U6(E) = \left\{ \begin{pmatrix} 0 & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right\}, \quad U7(E) = \left\{ \begin{pmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} \right\} \text{ and}$$

$$U8(E) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \gamma \end{pmatrix} \right\}$$

have nilpotent commutators with index of nilpotency ≤ 3.

Proof. Considering $U6(E)$ for $A_i = \begin{pmatrix} 0 & \beta_i & 0 \\ 0 & \alpha_i & 0 \\ 0 & 0 & \gamma_i \end{pmatrix}$, $i = 1, 2$, we have that

$$[A_1, A_2] = \begin{pmatrix} 0 & \beta_1\alpha_2 - \beta_2\alpha_1 & 0 \\ 0 & [\alpha_1, \alpha_2] & 0 \\ 0 & 0 & [\gamma_1, \gamma_2] \end{pmatrix}.$$

The only nonzero entry of $[A_1, A_2]^2 = [a_{ij}]$ is $a_{12} = (\beta_1\alpha_2 - \beta_2\alpha_1)[\alpha_1, \alpha_2]$. As $[x, y]^2 = 0$ we get that $[A_1, A_2]^3 = 0$. $\square$

Theorem 2. *All matrices of the subalgebras*

$$U9(E)=\left\{\begin{pmatrix} 0 & \dots & \dots & \dots & \dots & 0 & a_{1,n-1} & a_{1n}\\ 0 & a_{22} & 0 & \dots & \dots & \dots & 0 & a_{2n}\\ 0 & 0 & a_{33} & 0 & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \dots & \dots & \dots & \dots & 0 & a_{n-1,n-1} & 0\\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}\right\}$$

and

$$U10(E)=\left\{\begin{pmatrix} a & a_{12} & a_{13} & \dots & \dots & \dots & a_{1,n-1} & a_{1n}\\ 0 & a & 0 & \dots & \dots & \dots & 0 & a_{2n}\\ 0 & 0 & a & 0 & \dots & \dots & 0 & a_{3n}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \dots & \dots & \dots & \dots & 0 & a & a_{n-1,n}\\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a \end{pmatrix}\right\}$$

have nilpotent commutators with index of nilpotency ≤ 3.

Proof. Due to Proposition 4 for any $A, B \in U9(E)$ we get that the only nonzero entries in $[A,B]^2=[c_{ij}]$ are

$$\begin{aligned} c_{1,n-1} &= (\dots)[a_{n-1,n-1}, b_{n-1,n-1}],\\ c_{2n} &= [a_{22}, b_{22}](\dots). \end{aligned}$$

Thus in $[A,B]^3=[d_{ij}]$ we have $d_{1,n-1}=c_{1,n-1}[a_{n-1,n-1}, b_{n-1,n-1}]=0$ and all other entries are zero as well.

Considering $U10(E)$ let

$$A=\begin{pmatrix} a & a_{12} & a_{13} & \dots & \dots & \dots & a_{1,n-1} & a_{1n}\\ 0 & a & 0 & \dots & \dots & \dots & 0 & a_{2n}\\ 0 & 0 & a & 0 & \dots & \dots & 0 & a_{3n}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & \dots & \dots & \dots & \dots & 0 & a & a_{n-1,n}\\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a \end{pmatrix}$$

and

$$B = \begin{pmatrix} b & b_{12} & b_{13} & \dots & \dots & \dots & b_{1,n-1} & b_{1n} \\ 0 & b & 0 & \dots & \dots & \dots & 0 & b_{2n} \\ 0 & 0 & b & 0 & \dots & \dots & 0 & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & b & b_{n-1,n} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & b \end{pmatrix}.$$

In $[A,B]=[c_{ij}]$ we have that

$$\begin{aligned} c_{1k} &= [a,b_{1k}]+[a_{1k},b],\ k=2,\dots,n-1, \\ c_{kn} &= [a,b_{kn}]+[a_{kn},b],\ k=2,\dots,n-1, \\ c_{1n} &= [a,b_{1n}]+[a_{12},b_{2n}]+\dots+[a_{1,n-1},b_{n-1,n}]+[a_{1n},b] \\ c_{11} &= c_{22}=\dots=c_{nn}=[a,b]. \end{aligned}$$

According to Proposition 3 the only nonzero entry in $[A,B]^2=[d_{ij}]$ is

$$\begin{aligned} d_{1n} &= [a,b]([a_{12},b_{2n}]+[a_{13},b_{3n}]+\dots+[a_{1,n-1},b_{n-1,n}]) \\ &\quad +[a,b_{12}][a_{2n},b]+[a_{12},b][a,b_{2n}]+\dots \\ &\quad +[a,b_{1,n-1}][a_{n-1,n},b]+[a_{1,n-1},b][a,b_{n-1,n}] \\ &\quad +([a_{12},b_{2n}]+[a_{13},b_{3n}]+\dots+[a_{1,n-1},b_{n-1,n}])[a,b]. \end{aligned}$$

Applying Proposition 3 we get that $c_{11}d_{1n}=0$, thus $[A,B]^3=0$. $\square$

Remark 2. For the subalgebras

$$\left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & \dots & \dots & a_{1,n-1} & a_{1n} \\ 0 & a & 0 & \dots & \dots & \dots & 0 & a_{2n} \\ 0 & 0 & a & 0 & \dots & \dots & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & a & a_{n-1,n} \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \right\}$$

and

$$\left\{\begin{pmatrix} a_{11} & a_{12} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & a_{22} & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & a_{33} & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{nn} \end{pmatrix}\right\}$$

the index of nilpotency of the commutators in them is ≤ 4.

Proof. For the matrices A and B in the first case with entries a_1, a_{1i}, a_{in} and a_2, b_{1i}, b_{in}, respectively, we get that in $[A,B]^2 = [c_{ij}]$ the nonzero elements are $c_{1i} = (\dots)[a_1, a_2]$, $c_{in} = [a_1, a_2](\dots)$ for $i = 2, \dots, n-1$. Thus in $[A,B]^4 = [d_{ij}]$ the summands of the only nontrivial entry d_{1n} contain $[a_1, a_2]^2$, thus $[A,B]^4 = 0$.

For two matrices A and B in the second case the only nonzero entries in $[A,B]^2 = [c_{ij}]$ are c_{12} and $c_{n-1,n}$, thus $[A,B]^4 = 0$.

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REFERENCES

[1] A. Berele, A. Regev. Exponential growth for codimensions of some P.I. algebras. *J. Algebra* **241** (2001), 118–145.

[2] F. A. Berezin, M. S. Marinov. Particle spin dynamics as the Grassmann variant of classical mechanics. *Ann. Physics* **104**, *2* (1977), 336–362.

[3] V. Drensky. Free Algebras and PI-Algebras. Springer-Verlag Singapore, Singapore, 2000.

[4] V. Drensky, E. Formanek. Polynomial Identity Rings. Advanced Courses in Mathematics – CRM Barcelona, Basel, Birkhäuser, 2004.

[5] J. Frank. Fermion coherent states and Grassmann algebra. `www.theo3. physik.uni-stuttgart.de/lehre/ss09/hauptseminar/talks /fermions1.pdf`.

[6] A. Giambruno, D. La Mattina, V. M. Petrogradsky. Matrix algebras of polynomial codimension growth. *Israel J. Math.* **158** (2007), 367–378.

[7] A. Giambruno, M. Zaicev. Polynomial Identities and Assymptotic Methods. Math. Surveys and Monographs, vol. **122**, Providence, RI: American Mathematical Society, 2005.

[8] A. R. Kemer. Ideals of Identities of Associative Algebras. Translation of Mathematical Monographs, vol. **87**, Providence, RI: American Mathematical Society, 1991.

[9] D. Krakowski, A. Regev. The polynomial identities of the Grassmann algebra. *Trans. Amer. Math. Soc.* **181** (1973), 429–438.

[10] D. La Mattina, P. Misso. Algebras with involution with linear codimension growth. *J. Algebra* **305** (2006), 270–291.

[11] A. Popov. On the minimal degree identities of the matrices over the Grassmann algebra. American University of Blagoevgrad, preprint, 1997.

[12] K. Scharnhorst. A Grassmann integral equation. *J. Math. Physics* **44**, *11* (2003), 5415–5449.

[13] J. Szigeti. New determinants and the Cayley-Hamilton theorem for matrices over Lie nilpotent rings. *Proc. Amer. Math. Soc.* **125** (1997), 2245–2254.

[14] U. Vishne. Polynomial identities of $M_2(G)$. *Comm. Algebra* **30**, *1* (2002), 443–454.

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